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H. LUDESCHER & J. DE VRIES

ON A SUFFICIENT CONDITION FOR THE EXISTENCE OF G-COMPACTIFICATIONS

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On a sufficient condition for the existence of G-compactifications*)

by

H. Ludescher**) & J. de Vries

ABSTRACT

In this paper it is shown that if a topological transformation group is pointwise equicontinuous with respect to some uniformity of the phase space, then it can equivariantly be embedded in a topological transformation group with a compact phase space and the same acting group.

KEY WORDS & PHRASES: *topological transformation group, compactification, (uniform) equicontinuity*

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*) This report will be submitted for publication elsewhere.

**) Polytechnical Institute Timisoara, Computing Centre, Romania.

INTRODUCTION

In this paper we show that if the transition group of a topological transformation group is pointwise equicontinuous with respect to some uniform structure of the phase space, then the topological transformation group has a G-compactification (i.e. it can be embedded isomorphically in a topological transformation group with compact phase-space). This result is not covered by [6], where it has been shown that every topological transformation group on a completely regular space and with a locally compact phase group G has a G-compactification. As a consequence of our result we get that each topological transformation group with compact phase group G admits a G-compactification, a result which also follows from [6].

1. PRELIMINARIES

A *topological transformation group* (*ttg*) is a triple (G,X,π) where G is a topological group, X a topological space and $\pi : G \times X \to X$ is a continuous map, satisfying the following conditions:

(i) $\pi(e,x) = x$ for every $x \in X$ (e denotes the unit of G),

(ii) $\pi(t_1,\pi(t_2,x)) = \pi(t_1 t_2,x)$ for every t_1, $t_2 \in G$ and $x \in X$.

For a given ttg (G,X,π) we shall define for every $t \in G$ and every $x \in X$ the applications π^t resp. π_x by the formula $\pi^t(x) := \pi(t,x) =: \pi_x(t)$. Every π^t is called a *transition* and every π_x is called a *motion* of the ttg (G,X,π). We recall that the group $\{\pi^t | t \in G\}$ is a subgroup of the group $H(X)$ of all the authomeomorphisms of the space X and the map $t \mapsto \pi^t$ (called also the *natural homomorphism* associated to (G,X,π)) is a group-homomorphism. We shall say that the ttg (G,X,π) is *effective* if for each $t \in G \setminus \{e\}$ there is $x \in X$ with $\pi(t,x) \neq x$.

1.1. LEMMA. *Let* (G,X,π) *be a ttg with* X *Hausdorff and let* $H_\pi := \{t | \pi(t,x) = x$ *for every* $x \in X\}$. *Then:*

(a) H_π *is a closed invariant subgroup of* G,

(b) *Putting* $\tilde{G} = G/H_\pi$ *and* $\tilde{\pi}(tH_\pi,x) := \pi(t,x)$ *for* $t \in G$ *and* $x \in X$, *the triple* $(\tilde{G},X,\tilde{\pi})$ *is an effective ttg.*

(c) $\{\pi^t | t \in G\} = \{\tilde{\pi}^{\tilde{t}} | \tilde{t} \in \tilde{G}\}$.

PROOF. (a) and (b) follow from remark 1.3 in [3], and (c) is an immediate consequence of the definition of $\tilde{\pi}$. We notice here also that (G,X,π) is effective iff $t \mapsto \pi^t$ is injective or iff $H = \{e\}$; in this case (G,X,π) and $(\tilde{G},X,\tilde{\pi})$ are in fact identical. □

1.2. Let (G,X,π) and (G,Y,σ) be two ttg's; a continuous function $f : X \to Y$ is called a *homomorphism* (of ttg's) if $f(\pi(t,x) = \sigma(t,f(x))$ for every $(t,x) \in G \times X$. If Y is compact Hausdorff and f a topological embedding, then (G,Y,σ) is called a G-*compactification of* (G,X,π).

It is obvious that a ttg (G,X,π) can have a G-compacification only if X is completely regular (i.e. uniformizable and Hausdorff); this condition imposed on X will be considered henceforth as being automatically fullfilled. In [5] (see Theorem 7.3.12) the following fundamental result is proved concerning G-compactifications. It is a generalization of a result of R.B. Brook [2].

1.3. PROPOSITION. *The ttg* (G,X,π) *admits a* G-*compactification iff there is a uniformity* $\mathcal{U}$ *compatible with the topology of* X *so that for every* $U \in \mathcal{U}$ *there is a neighbourhood* V *of* e *with:*

$$t \in V \Rightarrow (\pi(t,x),x) \in U, \quad \textit{for each } x \in X. \quad \square$$

A ttg satisfying the above condition is said to be $\mathcal{U}$-*bounded*. It is obvious that the $\mathcal{U}$-boundedness of (G,X,π) is equivalent with the $\mathcal{U}$-equicontinuity of the family $\{\pi_x \mid x \in X\}$ at e.

We recall here the terminology concerning equicontinuity resp. uniform equicontinuity. Let X be a topological space, let $(Y,\mathcal{V})$ be a uniform space and let $F \subseteq Y^X$. We say that F is $\mathcal{V}$-*equicontinuous* at $x_0 \in X$ if for each $W \in \mathcal{V}$ there is a neigbourhood Γ of x_0 in X with $x \in \Gamma \Rightarrow (f(x),f(x_0)) \in W$, for every $f \in F$. The set F is said to be (*pointwise*) $\mathcal{V}$-*equicontinuous on* X if it is $\mathcal{V}$-equicontinuous at every $x \in X$. Finally, assuming that $(X,\mathcal{U})$ is a uniform space, F is said to be $(\mathcal{U},\mathcal{V})$-*uniformly equicontinuous* if for each $W \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(x,y) \in U \Rightarrow (f(x),f(y)) \in W$ for every $f \in F$.

1.4. LEMMA. *Let* (G,X,π) *be a ttg,* Φ *its transition group and* $\phi : G \to \Phi$ *the natural homomorphism associated to* (G,X,π). *Further let* $\mathcal{U}$ *be a uniformity of* X *and let us consider on* Φ *the topology of* $\mathcal{U}$*-convergence. Then* (G,X,π) *is* $\mathcal{U}$*-bounded iff* ϕ *is continuous at* e.

PROOF. As is well-known, the uniformity of $\mathcal{U}$-convergence on Φ is defined by the base $\{W(U) \mid U \in \mathcal{U}\}$, where

$$W(U) := \{(\phi,\psi) \mid \phi, \psi \in \Phi, (\phi(x), \psi(x)) \in U \text{ for all } x \in X\};$$

the corresponding uniform topology is called the topology of $\mathcal{U}$-convergence on Φ.

The continuity of ϕ at e can be written as follows: for each symmetric $U \in \mathcal{U}$ there is a neighbourhood V of e in G so that:

$$t \in V \Rightarrow \phi(t) \in W(U)[\phi(e)]$$

that is,

$$t \in V \Rightarrow (\phi(e), \phi(t)) \in W(U)$$

or equivalently,

$$(\pi^t(x),x) \in U \text{ for all } x \in X \text{ and } t \in V. \qquad \square$$

2. MAIN RESULT

We begin this section with a lemma which has some intrinsic interest.

2.1. LEMMA. *Let* $\mathcal{U}$ *be a uniformity of the topological space* X *and let* $\Phi \subseteq X^X$ *be a semigroup (by composition), which contains* 1_X *(the identity on* X*). If* Φ *is pointwise* $\mathcal{U}$*-equicontinuous on* X*, then there is a uniformity* $\mathcal{V}$ *of* X *making* Φ $(\mathcal{V},\mathcal{V})$*-uniformly equicontinuous.*

PROOF. For each $U \in \mathcal{U}$, define $\Phi(U) := \{(x,y) \mid (\phi(x),\phi(y)) \in U \text{ for every } \phi \in \Phi\}$. We have for every $U, V \in \mathcal{U}$:

(a) $\Phi(U) \supseteq \Delta(X)$ ($\Delta(X)$ denotes the diagonal of $X \times X$),
(b) $\Phi(U \cap V) = \Phi(U) \cap \Phi(V)$,
(c) $[\Phi(U)]^{-1} = \Phi(U^{-1})$,
(d) $[\Phi(U)]^2 \subseteq \Phi(U^2)$.

The relations (a) - (d) show that $\{\Phi(U) \mid U \in \mathcal{U}\}$ is a base for a uniformity on X; this uniformity will be designated by $\Phi(\mathcal{U})$. Since $1_X \in \Phi$ it follows that $\Phi(U) \subseteq U$ for every $U \in \mathcal{U}$, hence $\mathcal{U} \subseteq \Phi(\mathcal{U})$. In particular, the topology induced by $\Phi(\mathcal{U})$ is finer than the original topology on X.

Conversely, let us consider $U \in \mathcal{U}$ and $x_0 \in X$; by $\mathcal{U}$-equicontinuity of Φ there is a neighbourhood Γ of x_0 such that:

$$x \in \Gamma \Rightarrow \phi(x) \in U[\phi(x_0)] \quad \text{for every } \phi \in \Phi,$$

hence $(x_0,x) \in \Phi(U)$ and therefore $x \in \Phi(U)[x_0]$. In other words $\Gamma \subseteq \Phi(U)[x_0]$, that is $\Phi(U)[x_0]$ is a neighbourhood of x_0 in the original topology of X. This means that the topology induced by $\Phi(\mathcal{U})$ is weaker than that induced by $\mathcal{U}$. Consequently, $\Phi(\mathcal{U})$ is a uniformity of X.

The procedure applied to $\mathcal{U}$ can be repeated in the case of $\Phi(\mathcal{U})$ giving rise to the uniformity $\Phi(\Phi(\mathcal{U}))$. Because $1_x \in \Phi$ and because Φ is closed under composition we have for each $U \in \mathcal{U}$:

$$\Phi(\Phi(U)) = \{(x,y) \mid (\phi(x),\phi(y)) \in \Phi(U)\}$$

$$= \{(x,y) \mid (\psi(\phi(x)),\psi(\phi(y))) \in U \text{ for all } \phi,\psi \in \Phi\} = \Phi(U),$$

so $\Phi(\mathcal{U}) = \Phi(\Phi(\mathcal{U}))$. Obviously Φ is $(\Phi(\Phi(U)),\Phi(U))$-uniformly equicontinuous, i.e. Φ is $(\Phi(\mathcal{U}),\Phi(\mathcal{U}))$-uniformly equicontinuous. □

2.2. PROPOSITION. *Let* (G,X,π) *be a ttg and let* Φ *be its transition group. If* Φ *is* $\mathcal{U}$*-equicontinuous on* X *(with respect to a certain uniformity* $\mathcal{U}$ *of* X*), then* (G,X,π) *has a* G*-compactification.*

PROOF. By lemma 2.1 we can assume that Φ is $(\mathcal{U},\mathcal{U})$-uniformly equicontinuous. Let $\mathcal{U}_d$ be the right uniformity on G, i.e. the uniformity having as a base the sets $V_\Theta = \{(t,s) \mid st^{-1} \in \Theta\}$, where Θ is an arbitrary neighbourhood of e. Let us observe that

(2) $(t,s) \in V_\Theta \Rightarrow (tu,su) \in V_\Theta$ for each $u \in G$.

For each $(V,U) \in \mathcal{U}_d \times \mathcal{U}$ we define now the set

(3) $\bar{W}(V,U) := \{(\pi^t(x),\pi^s(y)) \mid (t,s) \in V \text{ and } (x,y) \in U\}$.

The following inclusions are immediate:

(B1) $\bar{W}(V,U) \supseteq \Delta(X)$,

(B2) $\bar{W}(V_1 \cap V_2, U_1 \cap U_2) \subseteq \bar{W}(V_1,U_1) \cap \bar{W}(V_2,U_2)$,

(B3) $\bar{W}(V,U)^{-1} \supseteq \bar{W}(V^{-1},U^{-1})$,

and they hold for every (V,U), (V_1,U_1), $(V_2,U_2) \in \mathcal{U}_d \times \mathcal{U}$.

If now $(V_0,U_0) \in \mathcal{U}_d \times \mathcal{U}$, we choose $(V,U_1) \in \mathcal{U}_d \times \mathcal{U}$ with $V^2 \subseteq V_0$ and $U_1^2 \subseteq U_0$. By the uniform equicontinuity of Φ there is $U \in \mathcal{U}$ with $(x,y) \in U \Rightarrow (\pi^t(x),\pi^t(y)) \in U_1$ for every $t \in G$. A straightforward reasoning using (2), shows that:

(B4) $\bar{W}(V,U)^2 \subseteq \bar{W}(V_0,U_0)$.

From (B1) - (B4) it follows that $\{\bar{W}(V,U) \mid V \in \mathcal{U}_d, U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{W}$ on X. In order to prove that $\mathcal{W}$ is a uniformity of X we shall denote the original topology on X by $\mathcal{T}_1$, whereas the topology induced by $\mathcal{W}$ will be denoted by $\mathcal{T}_2$.

The inequality $\mathcal{T}_2 \leq \mathcal{T}_1$ follows readily, observing that $U \subseteq \bar{W}(V,U)$ for $(V,U) \in \mathcal{U}_d \times \mathcal{U}$. We prove the converse inequality. To this end, let $x_0 \in X$ and let Γ be a $\mathcal{T}_1$-neighbourhood of x_0. By the continuity of π there is a neighbourhood Θ of e and a $\mathcal{T}_1$-neighbourhood Γ_1 of x_0 such that:

$$(t,x) \in \Theta \times \Gamma_1 \Rightarrow \pi(t,x) \in \Gamma.$$

As $\mathcal{T}_1$ is a uniform topology there is $U_1 \in \mathcal{U}$ such that $U_1[x_0] \subseteq \Gamma_1$. Now we choose $U \in \mathcal{U}$ having the property:

$$(x,y) \in U \Rightarrow (\pi^t(x),\pi^t(y)) \in U_1 \quad \text{for every } t \in G.$$

We will prove:

(4) $\bar{W}(V_\Theta,U)[x_0] \subseteq \Gamma$.

Let $y \in \bar{W}(V_\Theta,U)[x_0]$; then $(x_0,y) = (\pi^s(u),\pi^r(v))$, where $(s,r) \in V_\Theta$, and $(u,v) \in U$. But $(x_0,y) = (\pi^s(u),\pi^{rs^{-1}}(\pi^s(v)))$, and by the choice of U: $(\pi^s(u),\pi^s(v)) \in U_1$, hence $\pi^s(v) \in U_1[\pi^s(u)] = U_1[x_0] \subseteq \Gamma_1$. On the other hand $rs^{-1} \in \Theta$, so that $\pi(rs^{-1},\pi^s(v)) \in \Gamma$. But this means that $\pi^r(v) = y \in \Gamma$, i.e. (4) is proved. Consequently, Γ is also a T_2-neighbourhood of x_0, that is $T_1 \leq T_2$. The compatibility of $\mathcal{W}$ and T_1 is proved.

The uniformity of the $\mathcal{W}$-convergence on Φ has as base all the sets of the form

$$\lambda(V,U) := \{(f,g) \mid f,g \in \Phi, (f(x),g(x)) \in \bar{W}(V,U) \text{ for all } x \in X\},$$

with $(V,U) \in \mathcal{U}_d \times \mathcal{U}$.

We consider the natural homomorphism $\phi : G \to \Phi$ and $(V,U) \in \mathcal{U}_d \times \mathcal{U}$. If $(t,s) \in V$, then $(\phi(t),\phi(s)) = (\pi^t,\pi^s) \in \lambda(V,U)$, because $(\pi^t(x),\pi^s(x)) \in \bar{W}(V,U)$ for each $x \in X$. It follows that ϕ is uniformly continuous with respect to the uniformity $\mathcal{U}_d$ on G and the uniformity of $\mathcal{W}$-convergence on Φ. In particular, ϕ is continuous with respect to the corresponding topologies. By lemma 1.4. we get that (G,X,π) is $\mathcal{W}$-bounded. □

<u>2.3. COROLLARY</u>. *Let* (G,X,π) *be a ttg with* G *compact. Then* (G,X,π) *has at least one* G-*compactification.*

<u>PROOF.</u> We must find a uniformity $\mathcal{U}$ of X so that the transition group Φ of (G,X,π) is $\mathcal{U}$-equicontinuous on X. It is not a restriction if we assume that (G,X,π) is effective otherwise we consider $(\tilde{G},X,\tilde{\pi})$ defined in lemma 1.1. and notice that it is effective and has the same transition group as (G,X,π). In this case the natural homomorphism $\phi : G \to \Phi$ is a bijection.

Considering on Φ the image through ϕ of the topology of G we get that Φ is a compact topological group. Because π is continuous on $G \times X$ the function $\sigma : X \times \Phi \to X$, given by: $\sigma(x,f) = f(x)$, is continuous as well (we have: $\sigma(x,f) = \pi(\phi^{-1}(f),x)))$. Using a well known theorem about function spaces (e.g. [4], ch. VII) it follows that Φ is $\mathcal{U}$-equicontinuous on X for every uniformity $\mathcal{U}$ of the space X. □

2.4. EXAMPLE. Let us consider the ttg $(G, LUC_u(G), \tilde{\rho})$, where $LUC_u(G)$ is the space of all left-uniformly continuous real-valued functions on the topological group G endowed with the topology of the uniform convergence (on $\mathbb{R}$ we consider the usual uniformity). The action $\tilde{\rho}$ is defined by $\tilde{\rho}^t f(s) = f(st)$ for $f \in LUC_u(G)$ and $s, t \in G$. Then $\tilde{\rho}$ is continuous ([5], prop.2.2.4) and obviously $\{\tilde{\rho}^t | t \in G\}$ is uniformly equicontinuous with respect to the uniformity of uniform convergence. Thus $(G, LUC_u(G), \tilde{\rho})$ has a G-compactification.

The example given above shows us that our result is effectively not covered by [6], because we do not require the local compactness of G.

In the next example we point out that considering the uniformities $\Phi(\mathcal{U})$ (lemma 2.1.) resp. $\mathcal{W}$ (proposition 2.2.) is not superfluous; as we shall see the $\mathcal{U}$-equicontinuity of the transition group of a ttg generally does not imply the $(\mathcal{U},\mathcal{U})$-uniform equicontinuity, nor the $\mathcal{U}$-boundedness of the respective ttg.

2.5. EXAMPLE. Let $\mathbb{C}$ be the complex plane and let S^1 be the unit circle in $\mathbb{C}$. Further we consider $X_n := \{z | z \in C, |z| = 1+\frac{n}{n+1}\}$, $X := \bigcup_{n=1}^{\infty} X_n$ and define $\pi : S^1 \times X \to X$ as follows: $\pi(t,z) := t^n z$ if $(t,z) \in S^1 \times X_n$. Notice that $\{\pi^t | t \in S^1\}$ is $\mathcal{V}$-equicontinuous on X for every uniformity $\mathcal{V}$ of X (S^1 being compact). We shall consider on X the uniformity induced by the additive uniformity of $\mathbb{C}$; let us designate it by $\mathcal{U}_0$.

First, notice that $\Phi = \{\pi^t | t \in S^1\}$ is not $(\mathcal{U}_0,\mathcal{U}_0)$-uniformly equicontinuous, otherwise on Φ the topology of the $\mathcal{U}_0$-convergence would coincide with that of the pointwise convergence ([1], theorem 1, ch.X, §2). But this is not the case; indeed we choose $t_n := \exp\left(\frac{i\pi}{n}\right)$ we have $t_n \to 1$ i.e. $\pi^{t_n} \to \pi^1$ (pointwise). Taking now $z_n \in X_n$ we get $|\pi^{t_n}(z_n) - z_n| = |-2z_n| > 2$, hence π^{t_n} does not converge to π^1 in the topology of the $\mathcal{U}_0$-convergence. This implies that in our case $\mathcal{U}_0 \neq \Phi(\mathcal{U}_0)$. Finally (S^1,X,π) is not $\mathcal{U}_0$-bounded because in the contrary case lemma 1.4. would imply: $\pi^{t_n} \to \pi^1$ ($\mathcal{U}_0$-uniformly) which is obviously false. It follows that (S^1,X,π) is not $\Phi(\mathcal{U}_0)$-bounded, because $\mathcal{U}_0 \subsetneq \Phi(\mathcal{U}_0)$.

Hence in the given example the uniformities $\mathcal{U}_0$, $\Phi(\mathcal{U}_0)$ and $\mathcal{W}$ (build from $\Phi(\mathcal{U}_0)$, cf. prop.2.2) are pairwise distinct.

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